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STRESS SINGULARITY AT THE VERTEX OF A FLAT WEDGE-SHAPED CRACK BY A VARIATIONAL METHOD

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Three dimensional elasticity problems are generally complex. In this paper we present the analysis for the stress singularity at the apex of a three dimensional, flat, wedge-shaped crack under general loadings. The problem is reduced to a set of coupled dual integral equations. Because of the complexity they are not amenable to a closed form solution. A variational method is developed to handle such problems. The physical interpretation of the results is also presented.

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INTRODUCTION

The theory of fracture mechanics has been a very successful tool in engineering application in recent years. This is mainly due to the use of a single characteristic parameter namely the stress intensity factor, that is the coefficient of the stress singularity at the tip of a crack in the linear theory of elasticity. In most of the two dimensional cracks, in homogeneous media, the singularity is of the order one half. For the three dimensional cracks, however, the singularity depends upon the geometric configurations.

In this report we study the singularities at the apex of a thin wedge-shaped crack shown in Figure 1 under three loading conditions. Using the near field approach, the problem is reduced to an eigenvalue problem for coupled dual integral equations. The results indicate that cracks tend to straighten out at the apex.

In the following sections we first prove the variational theorem by which the eigenvalue of coupled integral equations is to be obtained. This eigenvalue problem is not the linear one commonly encountered in mathematical physics. Next we present Papkovitch stress function approach to three dimensional theory of elasticity. Then coupled dual series relations are obtained by using mixed boundary conditions. These series are transformed to coupled singular integral equations. Finally the variational method is applied to the coupled integral equations to obtain eigenvalues.

The variational method completely avoids the solution of complex singular integral equations. In this study Macsyma was found to be an indispensable tool at all levels of analysis.

VARIATIONAL THEOREM

Consider the following pair of homogeneous coupled integral equations with Fredholm kernels.

$$\int_0^\alpha K_{11}(\phi,\psi;\mu)f(\psi)d\psi + \int_0^\alpha K_{12}(\phi,\psi;\mu)g(\psi)d\psi = 0 \tag{1}$$

$$\int_0^\alpha K_{21}(\phi,\psi;\mu)f(\psi)d\psi \quad \int_0^\alpha K_{22}(\phi,\psi;\mu)g(\psi)d\psi = 0 \tag{2}$$

where eigenvectors f and g and the eigenvalue μ are unknown and $K_{11}(\phi,\psi;\mu)$ etc. involve μ in a linear or nonlinear fashion. Construct the following characteristic equation for the determination of μ^* with appropriate trial functions $f^*(\psi)$ and $g^*(\psi)$.

$$\left(\int_0^\alpha f^*(x)\int_0^\alpha K_{11}(x,y)f^*(y)dydx\right)\left(\int_0^\alpha g^*(\phi)\int_0^\alpha K_{22}(\phi,\psi)g^*(\psi)d\psi d\phi\right)$$

$$-\left(\int_0^\alpha f^*(x)\int_0^\alpha K_{12}(x,y)g^*(y)dydx\right)\left(\int_0^\alpha g^*(\phi)\int_0^\alpha K_{21}(\phi,\psi)f^*(\psi)d\psi d\phi\right) = 0 \tag{3}$$

If f^* and g^* vary around the exact solutions f and g as

$$f^*(\phi) = f(\phi) + \delta\xi(\phi) \quad , \quad f^*(\phi) = g(\phi) + \delta\eta(\phi) \tag{4}$$

then $(\mu^*-\mu)$ is stationary around δ as δ approaches zero. That is

$$\mu^* = \mu + 0(\delta^2)$$

provided

$$K_{11}(\phi,\psi) = K_{11}(\psi,\phi)$$
$$K_{22}(\phi,\psi) = K_{22}(\psi,\phi) \tag{5}$$
$$K_{12}(\phi,\psi) = K_{21}(\psi,\phi)$$

Proof: Symbolically we write equation (3) as

$$(f^*K_{11}f^*)(g^*K_{22}g^*) - (f^*K_{12}g^*)(g^*K_{21}f^*) = 0 \tag{6}$$

where

$$f^* = f + \delta\xi \quad , \quad g^* = g + \delta\eta \quad . \tag{7}$$

Expanding the kernels around μ, we have

$$K_{11}(\mu) = K_{11}(\mu^*) + \Delta K'_{11}(\mu^*) + 0(\Delta^2) \quad , \quad \text{etc.} \tag{8}$$

where

$$\Delta = \mu - \mu^* \tag{9}$$

Substituting from (7) and (8) into (6) and using (5) we obtain

$$\begin{aligned} &[(fK_{11}f)(gK_{22}g) - (fK_{12}g)(fK_{12}g)] \\ &+ 2\delta[(\xi K_{11}f)(gK_{22}g) - (\xi K_{12}g)(fK_{12}g) + (\eta K_{22}g)(fK_{11}f) - (fK_{12}\eta)(fK_{12}g)] \\ &+ L_1\Delta + L_2\delta^2 + \text{higher order terms} = 0 \end{aligned} \tag{10}$$

where

$$L_1 = (fK_{11}f)(gK'_{22}g) + (fK'_{11}f)(gK_{22}g) - 2(fK_{12}g)(fK'_{12}g) \tag{10a}$$

$$\begin{aligned} L_2 = &(\xi K_{11}\xi)(gK_{22}g) - (\xi K_{12}g)^2 + 4(fK_{11}\xi)(gK_{22}\eta) \\ &+ (fK_{11}f)(\eta K_{22}\eta) - (fK_{12}\eta)^2 - 4(fK_{12}\eta)(\xi K_{12}g) \quad . \end{aligned} \tag{10b}$$

Using equations (1) and (2) it is seen that the first two terms in (10) vanish.

Hence

$$\Delta = \delta^2(L_2/L_1) + 0(\delta^3) \quad . \tag{11}$$

The above proves that Δ is of the second order in δ. This completes the proof of the theorem. Because the eigenvalues involved in kernels can be in a nonlinear fashion we cannot prove its bounds as can be done for the linear case in Reference 1.

PAPKOVITCH STRESS FUNCTIONS

In the absence of body force the equation of equilibrium of a homogeneous isotropic elastic solid is given by

$$\nabla^2\bar{u} + \frac{1}{1-2\nu}\nabla\nabla\cdot\bar{u} = 0 \quad . \tag{12}$$

Using Helmholtz decomposition theorem the solution of (12) can be written as[2,3]

$$2G\bar{u} = \nabla B_o + \nabla(\bar{r}\cdot\bar{B}) - 4(1-\nu)\bar{B} \tag{13}$$

where G and ν are shear modulus and Poisson's ratio, B_o and $\bar{B}$ are known as Papkovitch stress functions satisfying

$$\nabla^2 B_o = 0 \quad , \quad \nabla^2\bar{B} = 0 \quad , \quad \bar{B} = \bar{i}\psi + \bar{j}\omega + \bar{k}\lambda \quad . \tag{14}$$

For computational purposes it is convenient to write the solution as a superposition of the following basic solutions.

$$\begin{array}{ll} \text{1st Basic solution:} & 2G\bar{u} = \nabla B_o \\ \text{2nd Basic solution:} & 2G\bar{u} = \nabla(x\psi) - 4(1-\nu)\psi\bar{i} \\ \text{3rd Basic solution:} & 2G\bar{u} = \nabla(y\omega) - 4(1-\nu)\omega\bar{j} \\ \text{4th Basic solution:} & 2G\bar{u} = \nabla(z\lambda) - 4(1-\nu)\lambda\bar{k} \end{array} \tag{15}$$

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[1] Barlett, C.C. and Noble, B., "A Variational Method for the Solution of Eigenvalue Problems Involving Mixed Boundary Conditions," Applied Science Research, Section B, Vol. 9, 1962.

[2] Green, A.E. and Zerna, W., THEORETICAL ELASTICITY, 2nd Edition, Oxford, 1968.

[3] Lure', A.T., THREE DIMENSIONAL PROBLEMS OF THE THEORY OF ELASTICITY, Inter-Science Publishers, 1964.

These basic solutions were transformed to the following spherical coordinate system

$$x = r \sin\theta \cos\phi \quad , \quad y = r \sin\theta \sin\phi \quad , \quad z = r \cos\theta \tag{16}$$

with the origin at the apex of the crack as shown in Figure 2. This enables us to obtain near field solutions and to study the stress singularity at the apex. The complete results of components of displacement and stress for basic solutions are given in Appendix 1. Since we are interested in the power singularity at the apex, so we choose Papkovitch potentials in the form

$$H(r,\theta,\phi) = r^{\mu} H_1(\theta,\phi) \tag{17}$$

where μ is the eigenvalue to be determined. As can be seen from Appendix 1, stress components will be of the form

$$\sigma = 0(r^{\mu-1}) \tag{18}$$

which will be singular when $\mu < 1$. For the displacements to be finite we seek positive eigenvalues between 0 and 1.

The near field geometry surrounding the apex permits us to write (17) as the separation of variables solution

$$H(r,\theta,\phi) = \sum_m r^{\mu} P_{\mu}^{m}(\cos\theta) \quad \begin{matrix} \cos m\phi \\ \text{or} \\ \sin m\phi \end{matrix} \tag{19}$$

where $P_{\mu}^{m}(x)$ is the associated Legendre function of the first kind with degree μ and order m. When (19) is substituted into the four basic solutions given in Appendix 1 we see that the forms of some solutions thus obtained are not convenient to work with. The final solutions used in this analysis are designated as solutions A, B, C, and D which are given in Appendix 2. Solution A was obtained by replacing μ by $\mu + 1$ after the substitution from (19)

into the first basic solution. Solution B is obtained by adding the second and the third basic solutions with proper trigonometric functions in (19), replacing m by m + 1 and then using Legendre recursion formulae. A similar process was used to obtain solution C. Solution D simply comes from the fourth basic solution with the use of (19).

Solutions A, B, C, and D must not be linearly independent. This is due to the fact that the condition that the vector point function is solenoidal has not been used explicitly in Papkovitch stress functions approach. We found these solutions are indeed not independent. A relation among them can be written symbolically in the form

$$(\mu+m+1)[C] = [B] + 2(\mu+m+1)[D] - 2(\mu-3+4\nu)[A] \quad . \tag{20}$$

Hereafter solution C is replaced by solutions A, B, D using (20).

THREE MODES OF CRACKS AND COUPLED INTEGRAL EQUATIONS

For a crack shown in Figure 2, the leading edges of the crack are $\phi = \pm\alpha$ and the crack is in the x-y plane ($\theta = \pi/2$). Let D^- and D^+ be the cracked and uncracked region of the plane $\theta = \pi/2$. Within the cracked region, the displacement is discontinuous. If the discontinuity is in the z-direction ($u_\theta^+ - u_\theta^-$ = finite), the crack is under mode I; if the discontinuity is in the x-direction ($u_x^+ - u_x^-$ = finite), the crack is defined to be under mode II; and if $u_y^+ - u_y^-$ = finite, the crack is defined to be under mode III. Boundary conditions for various modes are tabulated below.

TABLE I. BOUNDARY CONDITIONS ON $\theta = \pi/2$

	Non Mixed Conditions (in $D^- + D^+$)	Mixed Conditions in D^-	Mixed Conditions in D^+	
Mode I	$\tau_{\theta r} = \tau_{\theta\phi} = 0$	$\sigma_\theta = 0$	$u_\theta = 0$	(21)
Mode II	$\sigma_\theta = 0$	$\tau_{\theta r} = \tau_{\theta\phi} = 0$	$u_r = u_\phi = 0$	(22)
Mode III	$\sigma_\theta = 0$	$\tau_{\theta r} = \tau_{\theta\phi} = 0$	$u_r = u_\phi = 0$	(23)

For mode I, u_θ is even in ϕ. This leads to the use of the trigonometric functions at the top of (19). The boundary conditions of (22) and (23) are identical, but the symmetric properties are different for mode II and mode III. In the former case, u_r is even and u_ϕ is odd in ϕ while in the latter, the reverse is true. Hence the proper set of quantities should be selected in (19), for each case.

MODE I. Using equation (19) and the non-mixed conditions of (21), we have

$$B_m = 0 \quad , \quad A_m = (m+\mu+1)^{-1}(1-2\nu)D_m \quad . \tag{24}$$

The mixed boundary conditions of (21) and using (24) and (19), yield

$$\begin{aligned} &\sum b_m \cos m\phi = 0 \qquad 0 \leq \phi < \alpha \\ &\sum q_m b_m \cos m\phi = 0 \qquad \alpha < \phi \leq \pi \end{aligned} \tag{25}$$

where $\sum$ denotes the summation with respect to m for $m = 0,1,2,\ldots\infty$, and

$$b_m = (-m+\mu+1)D_m P^m_{\mu+1} \quad , \quad q_m = (-m+\mu+1)^{-1} P^m_\mu / P^m_{\mu+1} \quad , \quad P^m_\mu = P^m_\mu(0) \tag{26}$$

MODE II. For the homogeneous condition of σ_θ in (22), using (19), the coefficients A, B, and D are related in the form

$$2(1-\nu)D_m = (m+\mu+1)A_m + [\mu^2 + 2\nu(\mu+1) - 2(1-\nu)m - 1]B_m$$

This relation and the mixed conditions of (22) yield the following coupled dual series:

$$\sum E_m \cos m\phi = 0 \qquad 0 \leq \phi < \alpha$$

$$\sum (R_m E_m + S_m F_m)\cos m\phi = 0 \qquad \alpha < \phi \leq \pi \tag{27}$$

$$\sum{}' F_m \sin m\phi = 0 \qquad 0 \leq \phi < \alpha$$

$$\sum{}' (U_m F_m + T_m E_m)\sin m\phi = 0 \qquad \alpha < \phi \leq \pi \tag{28}$$

where $\sum{}'$ denotes the summation with respect to m for $m = 1,2,\ldots\infty$, and

$$E_m = -(m+\mu+1)\{\mu A_m + [4m(1-\nu)^2 + \mu(\mu+4\nu-m-3)]B_m\}P_\mu^m \tag{29}$$

$$F_m = (m+\mu+1)\{mA_m + [4\mu(1-\nu)^2 + m(\mu+4\nu-m-3)]B_m\}P_\mu^m \tag{30}$$

$$R_m = [m^2 - \mu(\mu+1)(1-\nu)]V_m \quad , \quad S_m = m(1-\nu-\nu\mu)V_m$$

$$U_m = [(1-\nu)m^2 - \mu(\mu+1)]V_m \quad , \quad T_m = m(1+\nu\mu)V_m \tag{31}$$

in (31) V_m stands for $P_{\mu+1}^m / P_\mu^m / [(m+\mu+1)(m^2-\mu^2)]$.

MODE III. Similar to the preceding case, we have

$$\sum{}' E_m \sin m\phi = 0 \qquad 0 \leq \phi < \alpha$$

$$\sum{}' (R_m E_m + S_m F_m)\sin m\phi = 0 \qquad \alpha < \phi \leq \pi \tag{32}$$

$$\sum F_m \cos m\phi = 0 \qquad 0 \leq \phi < \alpha$$

$$\sum (U_m F_m + T_m E_m)\cos m\phi = 0 \qquad \alpha < \phi \leq \pi \tag{33}$$

It was found that the dual series (25) for mode I is identical to that of a potential problem studied in References 4, 5, and 6 and will not be discussed here.

For convenience we make the following change of variables

$$\phi = \pi-\omega \quad , \quad \alpha' = \pi-\alpha \quad , \quad (-1)^m E_m = E_m' \quad , \quad (-1)^m F_m = F_m'$$
$$R_m = R_m'(1-\nu-\nu\mu) \quad , \quad S_m = S_m'(1-\nu-\nu\mu)$$
$$U_m = U_m'(1+\nu\mu)(-1) \quad , \quad T_m = T_m'(1+\nu\mu)(-1) \quad . \tag{34}$$

The dual series of (27), (28) now become

$$\sum E_m' \cos m\omega = 0 \tag{35}$$
$$\sum{}' F_m' \sin m\omega = 0 \qquad \alpha' < \omega \leq \pi \tag{36}$$
$$\sum (R_m'E_m'+S_m'F_m')\cos m\omega = 0 \tag{37}$$
$$\sum{}' (T_m'E_m'+U_m'F_m')\sin m\omega = 0 \qquad 0 \leq \omega < \alpha' \tag{38}$$

Let the right hand sides of (35) and (36) for the interval $0 < \omega < \alpha'$ be denoted by unknown functions $f(\omega)$ and $g(\omega)$, respectively. The Fourier inversion gives

$$E_o' = \frac{1}{\pi}\int_0^{\alpha'} f(\psi)d\psi \quad , \quad E_m' = \frac{2}{\pi}\int_0^{\alpha'} f(\psi)\cos m\psi d\psi \quad , \quad F_m' = \frac{2}{\pi}\int_0^{\alpha'} g(\psi)\sin m\psi d\psi \tag{39}$$

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[4]Noble, B., "The Potential and Charge Distribution Near the Tip of a Flat Angular Sector," EM-135, New York University, NY, 1959.

[5]Brown, S.N., and Stewartson, K., "Flow Near the Apex of a Plane Delta Wing," Journal of Institute of Mathematics and Its Applications, Vol. 5, p. 206, 1969.

[6]Morrison, J.A., and Lewis, J.A., "Charge Singularity at the Corner of a Flat Plate," SIAM, Journal of Applied Mathematics, Vol. 31, p. 233, 1976.

Substituting from (39) into (37), (38) and interchanging the order of summation and integration we have the following coupled integral equations for the determination of f and g,

$$\int_0^{\alpha'} K_{11}(\omega,\psi;\mu)f(\psi)d\psi + \int_0^{\alpha'} K_{12}(\omega,\psi;\mu)g(\psi)d\psi = 0 \tag{40}$$

$$\int_0^{\alpha'} K_{21}(\omega,\psi;\mu)f(\psi)d\psi + \int_0^{\alpha'} K_{22}(\omega,\psi;\mu)g(\psi)d\psi = 0 \tag{41}$$

where

$$\begin{aligned}
K_{11}(\omega,\psi;\mu) &= \frac{1}{2} R_o' + \sum{}' R_m' \cos m\omega \cos m\psi \\
K_{12}(\omega,\psi;\mu) &= \sum{}' S_m' \cos m\omega \sin m\psi \\
K_{21}(\omega,\psi;\mu) &= \sum{}' T_m' \sin m\omega \cos m\psi \\
K_{22}(\omega,\psi;\mu) &= \sum{}' U_m' \sin m\omega \sin m\psi \quad .
\end{aligned} \tag{42}$$

It can be shown that[7]

$$\frac{P^m_{\mu+1}}{P^m_\mu} = \frac{-2}{m-\mu-1} \Gamma\left(\frac{m-\mu+1}{2}\right)\Gamma\left(\frac{m+\mu+2}{2}\right)/\Gamma\left(\frac{m-\mu}{2}\right)/\Gamma\left(\frac{m+\mu+1}{2}\right) \quad . \tag{43}$$

Using (43) and (34) the following asymptotic expansions can be established.

$$R_m' = -(1-\nu-\mu\nu)^{-1}(1/m) + 0(1/m^2) \quad , \quad S_m' = 1/m^2 + 0(1/m^3)$$

$$U_m' = [-(1-\nu)/(1+\nu\mu)](1/m) + 0(1/m^2) \quad , \quad T_m' = 1/m^2 + 0(1/m^3) \quad . \tag{44}$$

Substituting from (44) into (42) and summing the dominant part of the series we have,[4,8]

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[4]Noble, B., "The Potential and Charge Distribution Near the Tip of a Flat Angular Sector," EM-135, New York University, NY.

[7]Magnus, W. and Oberhettinger, F., SPECIAL FUNCTIONS OF MATHEMATICAL PHYSICS, Chelsea Publishing Co., 1949.

[8]Jolly, L.B.W., SUMMATION OF SERIES, Dover Publications, p. 126, 1961.

$$K_{11}(\omega,\psi) = \frac{1}{1-\nu-\nu\mu}\,\frac{1}{2}\log 2\left|\cos\omega - \cos\psi\right| + \text{regular terms} \tag{45}$$

$$K_{22}(\omega,\psi) = \frac{-(1-\nu)}{2(1+\nu\mu)}\log\left|\sin\left(\frac{\omega+\psi}{2}\right)/\sin\left(\frac{\omega-\psi}{2}\right)\right| + \text{regular terms} \tag{46}$$

The previous expressions show the kernels have logarithmic singularities. Similar analysis can be carried out for equations (32) and (33) for mode III (interchanging R_m' and U_m', S_m' and T_m').

APPLICATION OF VARIATIONAL PRINCIPLE

Equations (40) and (41) are identical to equations (1) and (2) and all the conditions of the theorem required for the kernels are satisfied. In this section we shall apply the theorem to obtain approximate eigenvalues by assuming approximate trial functions. Without causing ambiguity we shall drop the asterisks and assume the following trial functions,

$$f(t) = (\beta_o + \cos t)\cos(t/2)(\cos t - \cos\alpha')^{1/2}$$

$$g(t) = [(1 - \cos\alpha') + 2\cos t]\sin(t/2)/(\cos t - \cos\alpha')^{1/2} \tag{47}$$

where

$$\beta_o = -\cos^2(\alpha'/2)\frac{1 + (1-\nu-\nu\mu)R_o' + \log\sin^2(\alpha'/2)}{(1-\nu-\nu\mu)R_o' + \log\sin^2(\alpha'/2)} \tag{48}$$

The above trial functions are the first approximations to the integral equations (40) and (41) with kernels (42) replaced by their dominant parts given by (44). The method of obtaining such solutions by direct computation is illustrated in Reference 9.

[9] Noble, B., Hussain, M.A., and Pu, S.L., "Apex Singularities for Corner Cracks Under Opening, Sliding, and Tearing Modes," to be published in the Proceedings of the International Conference on Fracture Mechanics in Engineering Application, Bangalore, India, 1979.

Substituting from (47) into the characteristic equation (3), changing the order of summation and integration and using the integral representation of Legendre functions,[7] we obtain

$$I_{11}I_{22} - I_{12}^2 = 0 \tag{49}$$

where

$$I_{11} = 2R_o'(1+2\beta_o+P_1)^2 + \sum{}' R_m'[P_{m+1} + (1+2\beta_o)(P_m+P_{m+1}) + P_{m-2}]^2$$

$$I_{22} = 4\sum{}' U_m'[P_{m+1} - P_1(P_m-P_{m-1}) - P_{m-2}]^2$$

$$I_{12} = -2\sum{}' S_m'[P_{m+1} + (1+2\beta_o)(P_m+P_{m-1}) + P_{m-2}][P_{m+1} - P_1(P_m-P_{m-1}) - P_{m-2}]$$

$$P_m = P_m(\cos\alpha') \tag{50}$$

For different values of ν and α' (the complementary angle to half of the vertex angle of the wedge-shaped crack) we get approximate values of μ from equation (49) for both modes II and III.

The form of (49) is very well suited for Macsyma evaluation and using eight terms for summations in (50), the results for μ are marked by x in Figure 3 where the solid lines are results obtained by another method.[9] The results by both methods are in good agreement.

A further refinement of results can be obtained by selecting

$$f(t) = (A + B\cos t)\frac{\cos(t/2)}{(\cos t - \cos\alpha')^{1/2}}$$

$$g(t) = (C + D\cos t)\frac{\sin(t/2)}{(\cos t - \cos\alpha')^{1/2}} \tag{51}$$

[7] Magnus, W. and Oberhettinger, F., SPECIAL FUNCTIONS OF MATHEMATICAL PHYSICS, Chelsea Publishing Co., 1949.

and formally extending the variational technique to a characteristic equation obtained from the vanishing of the determinant of a four by four system (A, B, C, D in (51) must not all vanish). The results, using Macsyma and summing to a maximum of eight terms in (50), are compared in the following table. They also are shown as · in Figure 3.

TABLE II. VALUES OF μ FOR $\nu = 0.25$

Mode	Half Vertex Angle α	Variational Method Using (47)	Variational Method Using (51)	Direct Method Ref. 9
II	0.1π	0.9333	0.9616	0.9582
	0.3π	0.6489	0.6961	0.6953
	0.5π	0.4752	0.5107	0.5017
	0.7π	0.3585	0.3749	0.3654
	0.9π	0.2213	0.2469	0.2137
III	0.1π	0.9739		
	0.3π	0.7642	0.8253	0.8270
	0.5π	0.4881	0.5192	0.5027
	0.6π	0.3829	0.4034	0.3914
	0.7π	0.3021		0.3015
	0.8π	0.2372		0.2335

[9] Noble, B., Hussain, M.A., and Pu, S.L., "Apex Singularities for Corner Cracks Under Opening, Sliding, and Tearing Modes," to be published in the Proceedings of the International Conference on Fracture Mechanics in Engineering Application, Bangalore, India, 1979.

Even the rigorous proof of the extension of the variational technique from the two by two system using trial functions (47) to the four by four system using trial functions (51) is still to be done, the results thus obtained are in good agreement with results achieved by using trial functions (47) or other methods.[9]

CONCLUSIONS

For modes II and III, the results show that the stress singularities are dominated by the vertex angle as well as the elastic constant ν of the material. The results further indicate that when the apex angle is greater than 180°, the stress singularity is stronger than one half enhancing the tendency of crack front to straighten out. Similarly, when the vertex angle is less than 180°, the stress singularity is less severe than one half and, again, this will tend to retard the growth at the vertex until the crack front straightens out.

Macsyma was extensively used throughout the analysis, especially in the generation and use of special functions such as Legendre functions, Gamma functions, Bessel functions from Share directory, in the summation of series, in the solution of linear equations, in seeking roots of determinants of matrices, in the plot routine and in the creation of file for Batch with Teco, etc. This investigation would have been extremely tedious without Macsyma. The methods as well as results in the full entirety, to our knowledge, do not seem to have appeared in literature.

[9] Noble, B., Hussain, M.A., and Pu. S.L., "Apex Singularities for Corner Cracks Under Opening, Sliding, and Tearing Modes," to be published in the Proceedings of International Conf. on Fracture Mechanics in Engr Application, Bangalore, India, 1979.

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APPENDIX 1

In this appendix we give components of displacement and stress in terms of Papkovitch stress functions ϕ, ψ, ω, and λ. A subscript to a stress function means the partial derivative of the stress function with respect to the variable represented by the subscript, e.g., $\phi_r = \partial\phi/\partial r$, $\phi_{rp} = \partial(\partial\phi/\partial r)/\partial p$. The variable $p = \cos\theta$ is used in the place of θ in both Appendix 1 and Appendix 2. The notation $\bar{p} = \sin\theta = (1-p^2)^{1/2}$ is also used.

The First Basic Solution:

$$2Gu_r = \phi_r$$

$$2Gu_\theta = -\bar{p}\phi_p/r$$

$$2Gu_\phi = \phi_\phi/(r\bar{p})$$

$$\sigma_r = \phi_{rr}$$

$$\sigma_\theta = r^{-2}(\bar{p}^2\phi_{pp} - p\phi_p + r\phi_r)$$

$$\sigma_\phi = r^{-2}(\phi_{\phi\phi}/\bar{p}^2 + r\phi_r - p\phi_p)$$

$$\tau_{\theta\phi} = -r^{-2}(\phi_{p\phi} + p\phi_\phi/\bar{p}^2)$$

$$\tau_{r\phi} = r^{-2}\bar{p}^{-1}(r\phi_{r\phi} - \phi_\phi)$$

$$\tau_{r\theta} = r^{-2}\bar{p}(-r\phi_{rp} + \phi_p)$$

The Second Basic Solution:

$$2Gu_r = [r\psi_r - (3-4\nu)\psi]\bar{p}\cos\theta$$

$$2Gu_\theta = -[\bar{p}^2\psi_p + (3-4\nu)p\psi]\cos\phi$$

$$2Gu_\phi = \psi_\phi\cos\phi + (3-4\nu)\psi\sin\phi$$

$$\sigma_r = [r\psi_{rr} - 2(1-\nu)\psi_r + 2\nu r^{-1}p\psi_p]\bar{p}\cos\phi + 2\nu(rp)^{-1}\psi_\phi\sin\phi$$

$$\sigma_\theta = r^{-1}[\bar{p}^2\psi_{pp} + (1-2\nu)p\psi_p + (1-2\nu)r\psi_r]\bar{p}\cos\phi + 2\nu(r\bar{p})^{-1}\psi_\phi \sin\phi$$

$$\sigma_\phi = (r\bar{p})^{-1}[\psi_{\phi\phi} + (1-2\nu)r\bar{p}^2\psi_r - (1-2\nu)p\bar{p}^2\psi_p]\cos\phi + 2(1-\nu)(r\bar{p})^{-1}\psi_\phi \sin\phi$$

$$\tau_{\theta\phi} = (-r\bar{p})^{-1}[\bar{p}^2\psi_{p\phi} + 2(1-\nu)p\psi_\phi]\cos\phi - (1-2\nu)r^{-1}\bar{p}\psi_p \sin\phi$$

$$\tau_{r\phi} = r^{-1}[r\psi_{\phi r} - 2(1-\nu)\psi_\phi]\cos\phi + (1-2\nu)\psi_r \sin\phi$$

$$\tau_{r\theta} = [-\bar{p}^2\psi_{rp} - (1-2\nu)p\psi_r + 2(1-\nu)r^{-1}\bar{p}^2\psi_p]\cos\phi$$

The Third Basic Solution:

$$2Gu_r = [r\omega_r - (3-4\nu)\omega]\bar{p}\sin\phi$$

$$2Gu_\theta = -[\bar{p}^2\omega_p + (3-4\nu)p\omega]\sin\phi$$

$$2Gu_\phi = \omega_\phi \sin\phi - (3-4\nu)\omega\cos\phi$$

$$\sigma_r = [r\omega_{rr} - 2(1-\nu)\omega_r + 2\nu r^{-1}p\omega_p]\bar{p}\sin\phi - 2\nu(r\bar{p})^{-1}\omega_\phi \cos\phi$$

$$\sigma_\theta = r^{-1}[\bar{p}^2\omega_{pp} + (1-2\nu)p\omega_p + (1-2\nu)r\omega_r]\bar{p}\sin\phi - 2\nu(r\bar{p})^{-1}\omega_\phi \cos\phi$$

$$\sigma_\phi = r^{-1}[\omega_{\phi\phi}/\bar{p}^2 + (1-2\nu)r\omega_r - (1-2\nu)p\omega_p]\bar{p}\sin\phi - 2(1-\nu)(r\bar{p})^{-1}\omega_\phi \cos\phi$$

$$\tau_{\theta\phi} = (-r\bar{p})^{-1}[\bar{p}^2\omega_{p\phi} + 2(1-\nu)p\omega_\phi]\sin\phi + (1-2\nu)r^{-1}\bar{p}\omega_p \cos\phi$$

$$\tau_{r\phi} = r^{-1}[r\omega_{r\phi} - 2(1-\nu)\omega_\phi]\sin\phi - (1-2\nu)\omega_r \cos\phi$$

$$\tau_{r\theta} = [-\bar{p}^2\omega_{rp} - (1-2\nu)p\omega_r + 2(1-\nu)r^{-1}\bar{p}^2\omega_p]\sin\phi$$

The Fourth Basic Solution:

$$2Gu_r = [r\lambda_r - (3-4\nu)\lambda]p$$

$$2Gu_\theta = [-p\lambda_p + (3-4\nu)\lambda]\bar{p}$$

$$2Gu_\phi = p\bar{p}^{-1}\lambda_\phi$$

$$\sigma_r = rp\lambda_{rr} - 2(1-\nu)p\lambda_r - 2\nu r^{-1}\bar{p}^2\lambda_p$$

$$\sigma_\theta = r^{-1}p\bar{p}^2\lambda_{pp} + (1-2\nu)p\lambda_r + r^{-1}(p^2+2\nu\bar{p}^2-2)\lambda_p$$

$$\sigma_\phi = r^{-1}[p\bar{p}^{-2}\lambda_{\phi\phi} + (1-2\nu)rp\lambda_r - (p^2+2\nu\bar{p}^2)\lambda_p]$$

$$\tau_{\theta\phi} = r^{-1}[-p\lambda_{p\phi} + (2-2\nu-\bar{p}^{-2})\lambda_{\phi}]$$

$$\tau_{r\phi} = (r\bar{p})^{-1}[rp\lambda_{\phi r} - 2(1-\nu)p\lambda_{\phi}]$$

$$\tau_{r\theta} = \bar{p}[-p\lambda_{rp} + 2(1-\nu)r^{-1}p\lambda_{p} + (1-2\nu)\lambda_{r}]$$

APPENDIX 2

In the following solutions, the selection of cos mϕ or sin mϕ depends on the geometry of the problem. The sign on top goes with the trigonometric function on the top and vice versa.

Solution [A]:

$$2Gu_r = (\mu+1)r^{\mu}P^{m}_{\mu+1}\begin{matrix}\cos m\phi\\ \sin m\phi\end{matrix}$$

$$2Gu_\theta = (r^{\mu}/\bar{p})[(\mu+1)pP^{m}_{\mu+1} - (m+\mu+1)P^{m}_{\mu}]\begin{matrix}\cos m\phi\\ \sin m\phi\end{matrix}$$

$$2Gu_\phi = \mp\ (r^{\mu}/\bar{p})mP^{m}_{\mu+1}\begin{matrix}\sin m\phi\\ \cos m\phi\end{matrix}$$

$$\sigma_r = \mu(\mu+1)r^{\mu-1}P^{m}_{\mu+1}\begin{matrix}\cos m\phi\\ \sin m\phi\end{matrix}$$

$$\sigma_\theta = (r^{\mu-1}/\bar{p}^2)\{m^2-\mu-1-\bar{p}^2\mu(\mu+1)]P^{m}_{\mu+1} + p(m+\mu+1)P^{m}_{\mu}\}\begin{matrix}\cos m\phi\\ \sin m\phi\end{matrix}$$

$$\sigma_\phi = (r^{\mu-1}/\bar{p}^2)[(-m^2+\mu+1)P^{m}_{\mu+1} - p(m+\mu+1)P^{m}_{\mu}]\begin{matrix}\cos m\phi\\ \sin m\phi\end{matrix}$$

$$\tau_{\theta\phi} = \mp\ (r^{\mu-1}/\bar{p}^2)m[\mu pP^{m}_{\mu+1} - (m+\mu+1)P^{m}_{\mu}]\begin{matrix}\sin m\phi\\ \cos m\phi\end{matrix}$$

$$\tau_{r\phi} = \mp\ (r^{\mu-1}/\bar{p})m\mu P^{m}_{\mu+1}\begin{matrix}\sin m\phi\\ \cos m\phi\end{matrix}$$

$$\tau_{r\theta} = (r^{\mu-1}/\bar{p})\mu[(\mu+1)pP^{m}_{\mu+1} - (m+\mu+1)P^{m}_{\mu}]\begin{matrix}\cos m\phi\\ \sin m\phi\end{matrix}$$

Solution [B]:

$$2Gu_r = r^{\mu}(\mu-3+4\nu)[(-m+\mu+1)P^{m}_{\mu+1} - (m+\mu+1)pP^{m}_{\mu}]\begin{matrix}\cos m\phi\\ \sin m\phi\end{matrix}$$

$$2Gu_\theta = (r^{\mu}/\bar{p})\{(-m-4+4\nu)(-m+\mu+1)pP^{m}_{\mu+1} + (m+\mu+1)[m+4-4\nu-\bar{p}^2(\mu+4-4\nu)]P^{m}_{\mu}\}\begin{matrix}\cos m\phi\\ \sin m\phi\end{matrix}$$

$$2Gu_\phi = \mp (r^\mu/\bar{p})(m+4-4\nu)[(-m+\mu+1)P^m_{\mu+1} - (m+\mu+1)pP^m_\mu] \begin{matrix}\sin m\phi\\ \cos m\phi\end{matrix}$$

$$\sigma_r = r^{\mu-1}[(-m+\mu+1)(\mu^2-3\mu+2\mu\nu-2m\nu-2\nu)P^m_{\mu+1} + (m+\mu+1)(-\mu^2+3\mu+2\nu)pP^m_\mu] \begin{matrix}\cos m\phi\\ \sin m\phi\end{matrix}$$

$$\sigma_\theta = (r^{\mu-1}/\bar{p}^2)\{(-m+\mu+1)[(-\mu^2-2\mu\nu-(3-2\nu)(m+1)\bar{p}^2 + (m^2+(5-4\nu)m+4-4\nu)]P^m_{\mu+1}$$

$$+ (m+\mu+1)p[-m^2-5m+4m\nu-4+4\nu+\bar{p}^2(\mu^2+3\mu-3-2\nu)]P^m_\mu\} \begin{matrix}\cos m\phi\\ \sin m\phi\end{matrix}$$

$$\sigma_\phi = (r^{\mu-1}/\bar{p}^2)\{(-m+\mu+1)[-m^2-5m+4m\nu-4+4\nu+\bar{p}^2(1-2\nu)(m+\mu+1)]P^m_{\mu+1}$$

$$+ (m+\mu+1)p[m^2+5m-4m\nu+4-4\nu-(1-2\nu)(2\mu+1)\bar{p}^2]P^m_\mu\} \begin{matrix}\cos m\phi\\ \sin m\phi\end{matrix}$$

$$\tau_{\theta\phi} = \pm (r^{\mu-1}/\bar{p}^2)\{(-m+\mu+1)p[m^2+(5-4\nu)m+4(1-\nu)]P^m_{\mu+1}$$

$$+ (m+\mu+1)[\bar{p}^2(m\mu+(3-2\nu)m+2(1-\nu)\mu+4(1-\nu))-m^2-(5-4\nu)m-4(1-\nu)]P^m_\mu\} \begin{matrix}\sin m\phi\\ \cos m\phi\end{matrix}$$

$$\tau_{r\phi} = \mp (r^{\mu-1}/\bar{p})[m(\mu-2+2\nu)+2\mu(1-\nu)-2(1-\nu)][(-m+\mu+1)P^m_{\mu+1} - (m+\mu+1)pP^m_\mu] \begin{matrix}\sin m\phi\\ \cos m\phi\end{matrix}$$

$$\tau_{r\theta} = (r^{\mu-1}/\bar{p})\{(-m+\mu+1)p[-m\mu+2(1-\nu)(m-\mu+1)]P^m_{\mu+1}$$

$$+ (m+\mu+1)[m\mu+2(1-\nu)(-m+\mu-1)+\bar{p}^2(-\mu^2+2-2\nu)]P^m_\mu\} \begin{matrix}\cos m\phi\\ \sin m\phi\end{matrix}$$

Solution [C]:

$$2Gu_r = r^\mu(\mu-3+4\nu)(-P^m_{\mu+1} + pP^m_\mu) \begin{matrix}\cos m\phi\\ \sin m\phi\end{matrix}$$

$$2Gu_\theta = (r^\mu/\bar{p})\{(-m+4-4\nu)pP^m_{\mu+1} + [(m-4+4\nu)+\bar{p}^2(\mu+4-4\nu)]P^m_\mu\} \begin{matrix}\cos m\phi\\ \sin m\phi\end{matrix}$$

$$2Gu_\phi = \mp (r^\mu/\bar{p})(m-4+4\nu)(-P^m_{\mu+1} + pP^m_\mu) \begin{matrix}\sin n\phi\\ \cos n\phi\end{matrix}$$

$$\sigma_r = r^{\mu-1}\{[-\mu^2+\mu(3-2\nu)-2m\nu+2\nu]P^m_{\mu+1} + (\mu^2-3\mu-2\nu)pP^m_\mu\} \begin{matrix}\cos m\phi\\ \sin m\phi\end{matrix}$$

$$\sigma_\theta = (r^{\mu-1}/\bar{p}^2)\{[\bar{p}^2(\mu^2+2\nu\mu-(3-2\nu)(m-1)) + (-m^2+(5-4\nu)m-4(1-\nu))]P^m_{\mu+1}$$

$$+ [\bar{p}^2(-\mu^2-3\mu-3+2\nu) + m^2-(5-4\nu)m+4(1-\nu)]pP^m_\mu\} \begin{matrix}\cos m\phi \\ \sin m\phi\end{matrix}$$

$$\sigma_\phi = (r^{\mu-1}/\bar{p}^2)\{[\bar{p}^2(1-2\nu)(m-\mu-1) + m^2-(5-4\nu)m+4(1-\nu)]P^m_{\mu+1}$$

$$+ [\bar{p}^2(1-2\nu)(2\mu+1) - m^2+(5-4\nu)m-4(1-\nu)]pP^m_\mu\} \begin{matrix}\cos m\phi \\ \sin m\phi\end{matrix}$$

$$\tau_{\theta\phi} = \pm (r^{\mu-1}/\bar{p}^2) [m^2-(5-4\nu)m+4(1-\nu)]pP^m_{\mu+1}$$

$$+ [\bar{p}^2(-m\mu-(3-2\nu)m+2(1-\nu)\mu+4(1-\nu))-m^2+(5-4\nu)m-4(1-\nu)]P^m_\mu\} \begin{matrix}\sin m\phi \\ \cos m\phi\end{matrix}$$

$$\tau_{r\phi} = \pm (r^{\mu-1}/\bar{p})[-m\mu+2(1-\nu)m+2(1-\nu)(\mu-1)](-P^m_{\mu+1} + pP^m_\mu) \begin{matrix}\sin m\phi \\ \cos m\phi\end{matrix}$$

$$\tau_{r\theta} = (r^{\mu-1}/p)\{[-m\mu+2(1-\nu)m+2(1-\nu)(\mu-1)]pP^m_{\mu+1}$$

$$+ [m\mu-2(1-\nu)m-2(1-\nu)(\mu-1)-p^2(-\mu^2+2-2\nu)]P^m_\mu\} \begin{matrix}\cos m\phi \\ \sin m\phi\end{matrix}$$

Solution [D]:

$$2Gu_r = (\mu-3+4\nu)r^\mu pP^m_\mu \begin{matrix}\cos m\phi \\ \sin m\phi\end{matrix}$$

$$2Gu_\theta = (r^\mu/\bar{p})\{(-m+\mu+1)pP^m_{\mu+1} + [\bar{p}^2(\mu+4-4\nu)-(\mu+1)]P^m_\mu\} \begin{matrix}\cos m\phi \\ \sin m\phi\end{matrix}$$

$$2Gu_\phi = \mp (mp/\bar{p})r^\mu P^m_\mu \begin{matrix}\sin m\phi \\ \cos m\phi\end{matrix}$$

$$\sigma_r = r^{\mu-1}\{2\nu(-m+\mu+1)P^m_{\mu+1} + (\mu^2-3\mu-2\nu)pP^m_\mu\} \begin{matrix}\cos m\phi \\ \sin m\phi\end{matrix}$$

$$\sigma_\theta = (r^{\mu-1}/\bar{p}^2)\{(m-\mu-1)[1-(3-2\nu)\bar{p}^2]P^m_{\mu+1} + [m^2+\mu+1+\bar{p}^2(-\mu^2-3\mu-3+2\nu)]pP^m_\mu\} \begin{matrix}\cos m\phi \\ \sin m\phi\end{matrix}$$

$$\sigma_\phi = (r^{\mu-1}/\bar{p}^2)\{(-m+\mu+1)[1+\bar{p}^2(-1+2\nu)]P^m_{\mu+1} + [-m^2-\mu-1+(1-2\nu)(2\mu+1)\bar{p}^2]pP^m_\mu\} \begin{matrix}\cos m\phi \\ \sin m\phi\end{matrix}$$

$$\tau_{\theta\phi} = \pm\ (r^{\mu-1}/\bar{p}^2)m\{(m-\mu-1)pP^m_{\mu+1} + [\mu+2-\bar{p}^2(\mu+3-2\nu)]P^m_\mu\}\begin{matrix}\sin m\phi\\ \cos m\phi\end{matrix}$$

$$\tau_{r\phi} = \pm\ (2-2\nu-\mu)mp(r^{\mu-1}/\bar{p})P^m_\mu\begin{matrix}\sin m\phi\\ \cos m\phi\end{matrix}$$

$$\tau_{r\theta} = (r^{\mu-1}/\bar{p})\{(-m+\mu+1)(\mu-2+2\nu)pP^m_{\mu+1} + [-\mu^2+\mu(1-2\nu)+2(1-\nu)$$

$$+ \bar{p}^2(\mu^2-2+2\nu)]P^m_\mu\}\begin{matrix}\cos m\phi\\ \sin m\phi\end{matrix}$$

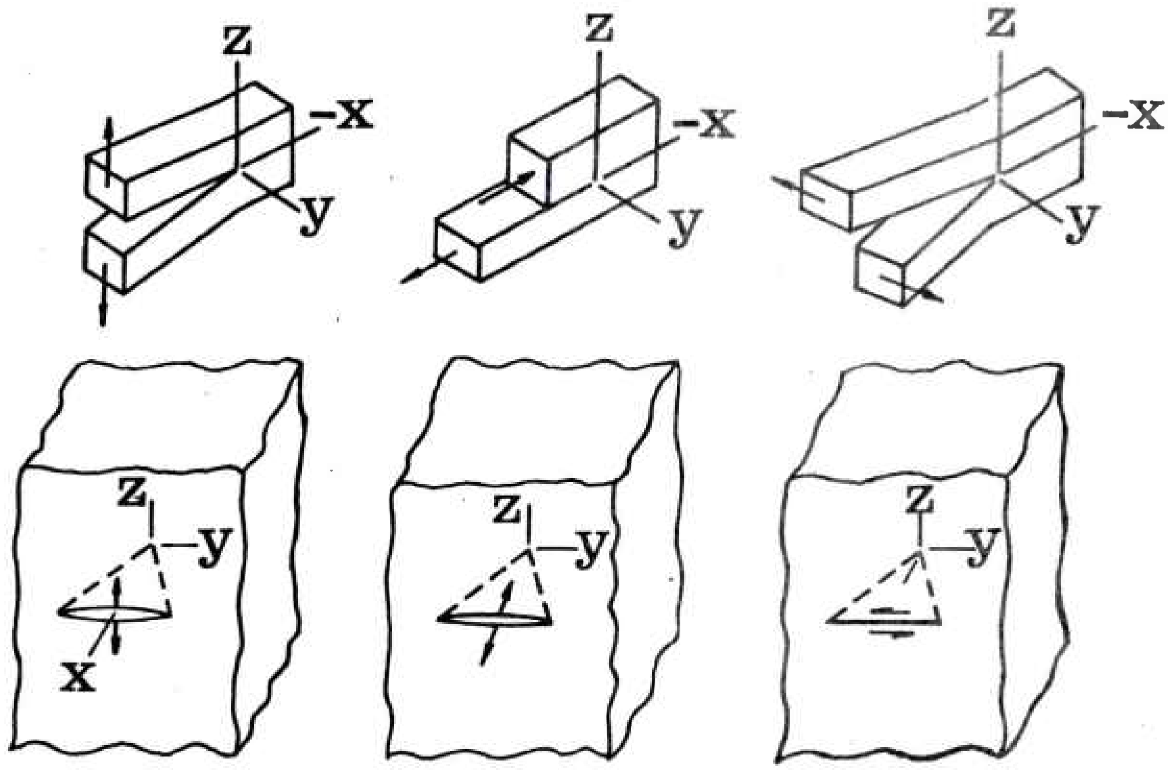

Figure 1. A flat wedge-shaped crack under three different modes and its two-dimensional counterparts.

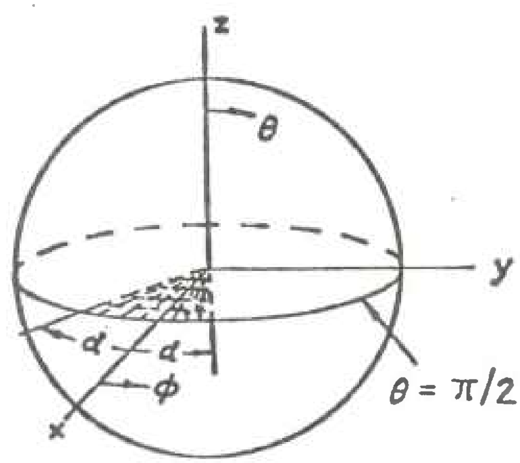

Figure 2. A spherical coordinate surrounding the apex of a thin angular sector crack.

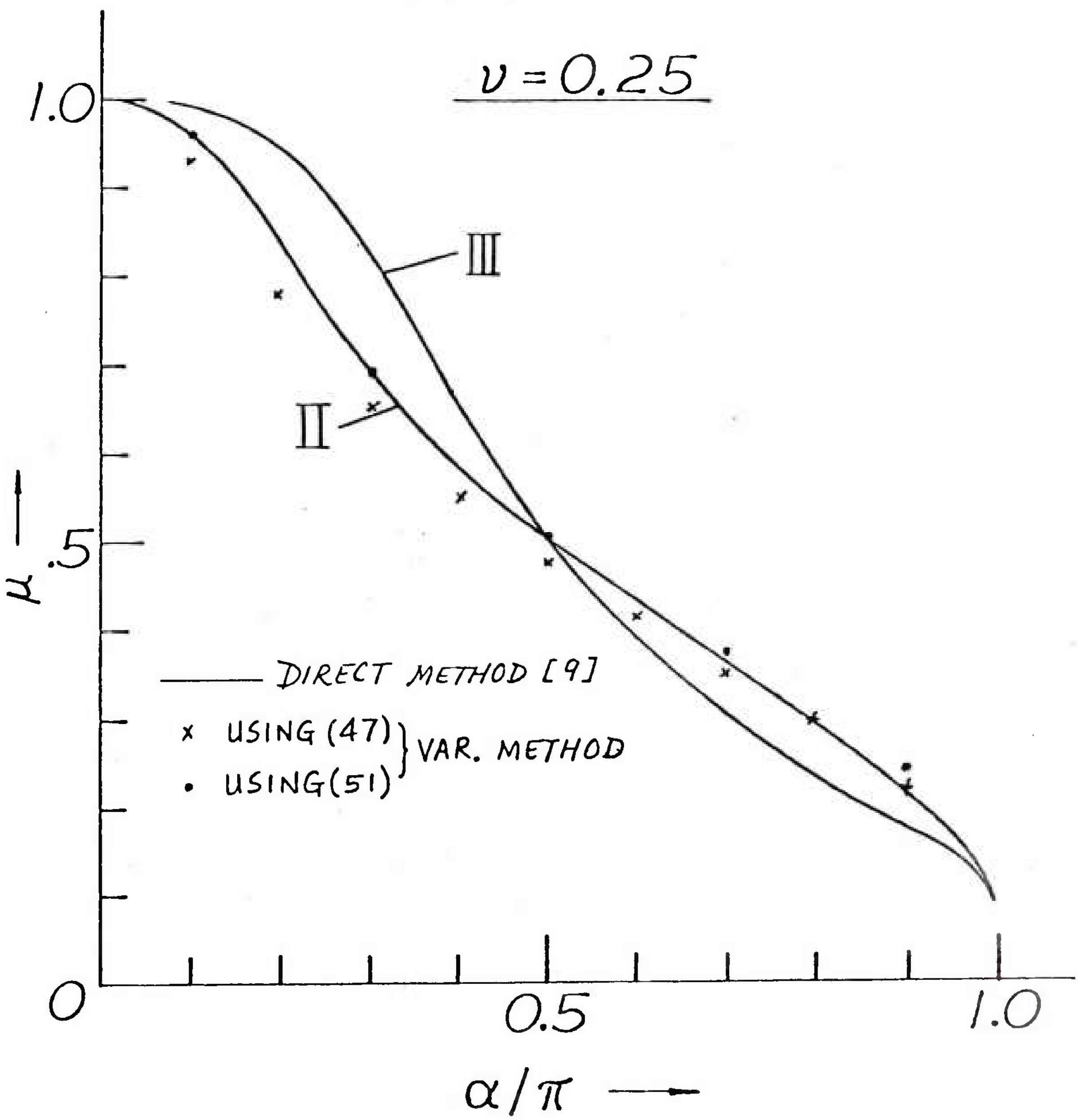

Figure 3. The eigenvalue μ as a function of α/π for ν = 0.25.

(a) Variational results indicated by x and · for mode II.

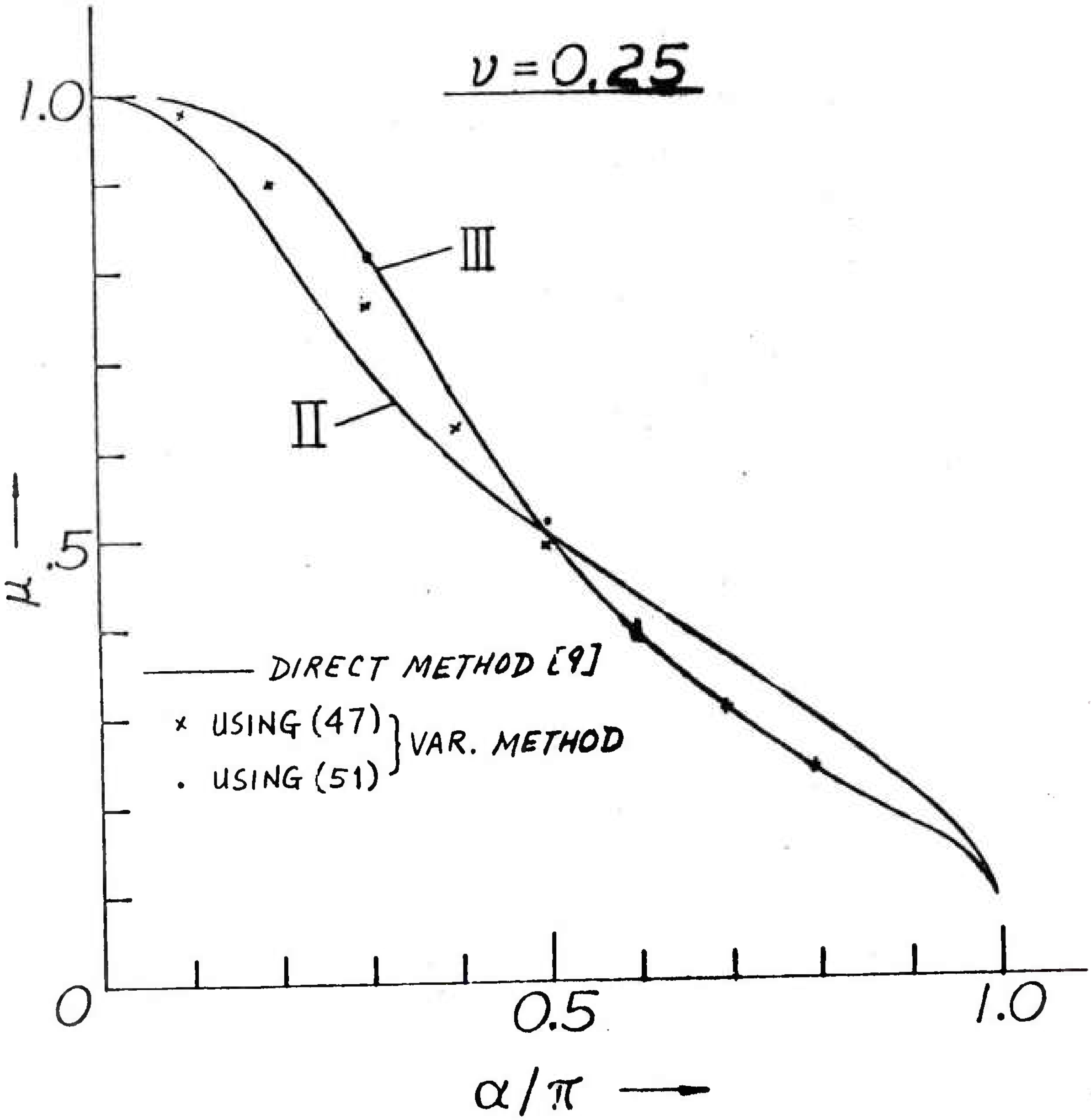

Figure 3(b). Variational results indicated by x and · for mode III.

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